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2T Physics, Scale Invariance and Topological Vector Fields

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Abstract We construct, in classical two-time physics, the necessary structure for the most general configuration space formulation of quantum mechanics containing gravity in $d+2$ dimensions. This structure is composed of a symmetric Riemannian metric tensor and of a vector field that defines a section of a flat $U(1)$ bundle over space-time. This construction is possible because of the existence of a finite local scale invariance of the Hamiltonian and because two-time physics contains, at the classical level, a local generalization of the discrete duality symmetry between position and momentum that underlies the structure of quantum mechanics.

Keywords Massless relativistic particles · Local scale invariance · 2T physics · Riemannian spaces

1 Introduction

The symmetry transformations of a classical action functional describing a physical system can be divided into four types. The most common type consists of the rigid (global) infinitesimal symmetry transformations. These are infinitesimal transformations of the dynamic variables, and possibly of the auxiliary variables that appear in the action functional, parametrized by constant arbitrary infinitesimal parameters.

Also quite common are the local infinitesimal symmetry transformations, which compose the second type. These are infinitesimal transformations of the dynamic variables, and auxiliary variables, parametrized by arbitrary infinitesimal parameters which depend on the manifold point where the transformations are performed.

The last two types of symmetry transformations are not infinitesimal transformations and correspond to the finite rigid and finite local symmetry transformations. These are in some cases rather subtle symmetry transformations because their finite character is related with topological aspects of the underlying manifold. And this relation is important because the topological aspects of a manifold are related to the non trivial diffeomorphic-covariant

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representations of the Heisenberg algebra over the manifold [1]. We can then use the finite symmetry transformations on a certain topologically non-trivial manifold to investigate the general structure of quantum mechanics.

In this paper we are interested in the Lorentz $SO(d,2)$ invariance. This invariance manifests itself as conformal invariance of the scalar relativistic massless particle in a d dimensional Minkowski space only if a compactification of the space-time is assumed [2, 3]. $SO(d,2)$ turns out to be the isometry of $(d+1)$-dimensional Anti de Sitter (AdS) space if a slightly different compactification of space-time is assumed [2]. These two slightly different compactifications then reveal that the d-dimensional Minkowski space is the border of the $(d+1)$-dimensional AdS space, an observation that is the cornerstone of the AdS/CFT conjecture [4, 5]. It is then important to understand other possible ramifications of the Lorentz $SO(d,2)$ invariance.

$SO(d,2)$ is also the rigid symmetry of two-time (2T) physics [6–12], where it appears as a consequence of the first class Hamiltonian constraints. From the point of view of 2T physics the known fundamental gravitational and gauge interactions in d dimensions are all embedded in a $d+2$ dimensional flat Minkowski space with two timelike dimensions. From this point of view the fundamental interactions display higher dimensional space-time symmetries that otherwise would remain hidden. In the current formulation of two-time physics compactification of the $(d+2)$-dimensional Minkowski space is avoided by considering only infinitesimal rigid $SO(d,2)$ transformations. In particular, only infinitesimal rigid scale and special conformal transformations are defined. The local versions of these two infinitesimal transformations, together with diffeomorphism invariance, compose the local $Sp(2,R) \sim SO(1,2)$ gauge invariance of two-time physics.

However, despite avoiding the consideration of the implications of space-time compactification, topological considerations necessarily arise in consequence of the nontrivial configuration space topology induced by the first class constraints of 2T physics. These topological considerations have a fundamental origin. Quantum dynamics requires the definition of a Riemannian metric structure on configuration space, whose determinant directly specifies the normalization of position eigenstates in order to ensure the correct covariant properties of the Heisenberg algebra representations under diffeomorphisms of the configuration manifold [1]. In addition, due to the local arbitrariness in the phase of position eigenstates,

$$|\tilde{x}\rangle = \exp\left\{\frac{i}{\hbar}\beta(x)\right\}|x\rangle \tag{1.1}$$

a flat $U(1)$ bundle is always associated to any such representation of the Heisenberg algebra [1]. In the case of a simply connected manifold, this flat $U(1)$ bundle may always be globally trivialized over the entire configuration manifold M, thereby corresponding to the ordinary trivial representation of the Heisenberg algebra. However, for configuration spaces of non trivial mapping class group $\pi_1(M)$, an infinity of inequivalent representations becomes possible, being labelled by the non trivial holonomies of the flat $U(1)$ bundle around the noncontractile cycles in the configuration manifold [1]. This last situation is exactly the one we have in 2T physics. This is because the Hamiltonian constraints require, for consistency, that the origin of phase space be removed. This induces a non-trivial configuration space topology which will then require the presence, in the quantized 2T theory, of a vector field of vanishing strength tensor associated to the flat $U(1)$ bundle which will characterize the inequivalent representations of the Heisenberg algebra. The search for a naturally induced metric structure in the $d+2$ dimensional space of 2T physics, and the construction of a classical 2T action with a background vector field of vanishing strength tensor, are the subjects of this paper.

A natural geometrical interpretation [21] of gauge fields is to identify the vector potentials A_M with the connection coefficients of the principal fiber space whose base is Riemannian space-time, the fiber being a finite gauge Lie group G. In this case, the stress tensor F_{MN} of the gauge field becomes the curvature tensor of the fiber space. For a flat $U(1)$ bundle, $F_{MN} = 0$. An approach to the introduction of background gravitational and gauge fields in 2T physics was first presented in [12]. In [12], the linear realization of the $Sp(2,R)$ gauge algebra of two-time physics is required to be preserved when background gravitational and gauge fields come into play. To satisfy this requirement, the gravitational field must satisfy a homothety condition [12], while in the absence of gravitational fields the gauge field $A_M(X)$ must satisfy the conditions [12]

$$X.A(X) = 0, \tag{1.2a}$$

$$\partial_M A^M(X) = 0, \tag{1.2b}$$

$$(X.\partial + 1)A_M(X) = 0 \tag{1.2c}$$

which were first proposed by Dirac [13] in 1936. Dirac proposed these conditions as subsidiary conditions to describe the usual 4-dimensional Maxwell electrodynamic theory as a theory in 6 dimensions which automatically displays $SO(4,2)$ symmetry.

Dirac's conditions (1.2) are a reflex of a hidden fundamental $Sp(2,R)$ symmetry in Maxwell's electrodynamics. This can be seen as follows. If we recall that in the topologically trivial [1] transition to quantum mechanics we can substitute $X^M \to X^M$ and $P_M \to i\hbar\frac{\partial}{\partial X^M}$, we can construct a semi-classical approximation where derivatives with respect to X^M are substituted by P_M and rewrite Dirac's conditions (1.2) in the form

$$X.A(X) = 0, \tag{1.3a}$$

$$P.A(X) = 0, \tag{1.3b}$$

$$(X.P + 1)A_M(X) = 0. \tag{1.3c}$$

For an electrodynamic vector field, the condition for the closure of the $Sp(2,R)$ gauge algebra of 2T physics is [12]

$$X^M F_{MN} = X^M\left(\frac{\partial A_N}{\partial X^M} - \frac{\partial A_M}{\partial X^N}\right) = 0. \tag{1.4}$$

Using again our semi-classical approximation, condition (1.4) becomes

$$(X.P + 1)A_N = P_N(X.A). \tag{1.5}$$

We see from (1.5) that the $Sp(2,R)$ closure condition (1.4) leads to Dirac's condition (1.3c) only if we first impose condition (1.3a). When this is done, the rigid $SO(4,2)$ invariance of electrodynamics is the reflex of a local $Sp(2,R)$ invariance. But for the case we are interested here, namely that of a topological vector field associated to a flat $U(1)$ bundle, $F_{MN} = 0$, and so condition (1.4) is trivially satisfied. Therefore, in the case of a topological vector field, even if we impose condition (1.3a) first, Dirac's condition (1.3c) can not be reached. On the contrary, if we impose (1.3c) first, then (1.3a) can not be reached. We must then search for an alternative set of conditions on the vector field if we want our topological 2T action to display local $Sp(2,R)$ and consequently rigid $SO(d,2)$ invariances.

It is important to find this correct set of subsidiary conditions on the topological vector field. As demonstrated in [1], vector fields of vanishing strength tensor play a fundamental role in the generalization of quantum mechanics in the position representation. In this generalization, this kind of vector field also necessarily appears, together with the determinant of the Riemannian metric tensor, in the most general expression of the position matrix elements for self-adjoint momentum operators in configuration spaces with non-trivial topology,

$$\begin{aligned}\langle X|\hat{P}_M|X'\rangle &= \frac{i\hbar}{G^{1/4}(X)}\frac{\partial}{\partial X^M}\left[\frac{1}{G^{1/4}(X)}\delta^d(X-X')\right]\\ &\quad+\frac{1}{\sqrt{G(X)}}A_M(X)\delta^d(X-X')\end{aligned} \tag{1.6}$$

where $G(X)=\det G_{MN}(X)$. We conclude from this generalization of quantum mechanics that vector fields of vanishing strength tensor in topologically non-trivial spaces will play an important role in the also important process of further unifying general relativity and quantum mechanics beyond the configuration space treatment exposed in [1].

However, as we saw above, the conditions (1.2) obtained in [12], or their semi-classical approximations (1.3), are not guaranteed to be valid for vector fields of vanishing strength tensor. Therefore these conditions can not be used in the process of accommodating gravity into quantum mechanics in the higher dimensional space-time of 2T physics. The set of conditions on the vector field we obtain in this paper has a different nature from that of the set (1.2). While the set (1.2) is formed from subsidiary kinematical conditions with no special significance, the set of conditions we obtain here has a more fundamental origin because it is formed by the first class Hamiltonian constraints for 2T physics with a topological vector field. This will be explicitly verified in section four, where we show that these first class constraints compose the correct conserved Hamiltonian Noether charge associated to local $Sp(2,R)$ invariance in the presence of the topological vector field.

The formulation of 2T physics with vector fields we present in this paper has implications, some of which are now being investigated, in the non-relativistic and relativistic quantum mechanics, in $d-1$ dimensions and d dimensions respectively, of physical systems enjoying local infinitesimal conformal $SO(1,2)\sim Sp(2,R)$ symmetry and/or global infinitesimal Lorentz $SO(d,2)$ symmetry. The list of these systems starts with the free massive non-relativistic particle and ends with black holes, passing through the harmonic oscillator, the Hydrogen atom, the de Sitter and Anti de Sitter spaces, and contains all the dynamic systems that have a unified description given by 2T physics. Our formulation of 2T physics with vector fields can also lead to interesting insights into the implications of the wave-particle duality on the general structure of quantum mechanics and provide a formulation of quantum mechanics with a single time where position and momentum are explicitly treated as locally indistinguishable variables [22]. Recall that the results in [1] are valid for configuration space only. Here the first class constraint structure of two-time physics, which is what ultimately requires a metric with two time-like dimensions, also requires the origin of phase space to be removed [3]. This creates a non-trivial phase space topology, with inequivalent diffeomorphic covariant representations of the Heisenberg algebra over the configuration space (viewed as part of the phase space), which are all classified in terms of vector fields with a vanishing second-rank antisymmetric strength tensor. In this paper we study this situation at a semi-classical level, present a Hamiltonian formulation of 2T physics with such a kind of vector fields, and show that the action we compute has a rigid infinitesimal $SO(d,2)$ invariance. We also show that our action has a local infinitesimal invariance which generalizes the local $Sp(2,R)$ invariance in the presence of the vector field and compute the corresponding conserved Hamiltonian Noether charge. These results can be used as the basis for a formulation of quantum mechanics which naturally accommodates gravity in higher dimensions based on the construction described in [1]. They also suggest that a momentum space version of the results in [1] is straightforward.

It has been known for some time [21] that the gravitational field, regarded as a gauge field, can correspond to several symmetry groups: (1) the general covariant group; (2) the local Lorentz group; and (3) the group of scale transformations of the interval. In the first case the properties of the gravitational field are determined by the properties of the metric tensor, and this gives the usual Einstein theory. In the second case, they are determined by the properties of the Ricci connection coefficients, and this leads to field equations of fourth order. In the third case it is assumed that the source of the field is the trace of the energy-momentum tensor and that the carriers are scalar particles [21]. Consequently, the approach based on gauge symmetries can lead to more general theories than that of Einstein. In this paper we are somewhat in the context of the third point of view. This is because Lorentz $SO(d,2)$ invariance manifests itself as conformal invariance of the relativistic scalar massless particle action in d dimensions, and the 2T physics action is the higher dimensional generalization of the scalar massless particle action to $d+2$ dimensions. For these reasons, in the next section we examine the rigid and local symmetries of the massless particle action. We present a finite local scale invariance of the particle's Hamiltonian that induces a transformation of the position coordinates which in this paper we are inclined to interpret as the classical correspondent of the quantum local phase transformation (1.1) of the position eigenstates. We also show how we can use this local scale invariance of the massless particle Hamiltonian to derive the classical analogues of the Snyder commutators [19], which were derived in 1947 in a projective geometry approach to the de Sitter space in the momentum representation.

In section three we review the construction of the 2T physics action and explicitly display its rigid and local infinitesimal symmetries. We compute the conserved Hamiltonian Noether charge and show that the finite local scale invariance we found for the massless particle has a simple and natural extension in 2T physics. Then we show how we can use this finite local scale invariance of the 2T Hamiltonian to induce a Riemannian metric structure in $d+2$ dimensions.

In section four we construct an action functional for 2T physics in the background of a vector field of vanishing strength tensor. We display its rigid infinitesimal Lorentz $SO(d,2)$ invariance and compute the conserved Hamiltonian Noether charge in the presence of the vector field. We find that this conserved charge is composed of the original first class constraints of 2T physics complemented with three first class constraints which involve the vector field and the canonical variables. These last three first class constraints must be used in the place of conditions (1.3) when $F_{MN}=0$. We also show how the Riemannian metric structure we found in the absence of the vector field in section three is preserved in the presence of the vector field. We conclude that we have found, already in classical 2T physics, the fundamental necessary ingredients for the topologically non trivial construction of quantum mechanics described in [1], that is, a Riemannian metric structure and a vector field of vanishing strength tensor. We were able to do this because 2T physics has a classical local $Sp(2,R)$ invariant generalization of the discrete duality symmetry between position and momentum that underlies the structure of quantum mechanics. Other concluding remarks appear in section five.

2 Massless Relativistic Particles

Before considering topological aspects in 2T physics, it is instructive to consider these aspects in massless scalar particle theory. A massless scalar relativistic particle in a d-dimensional Minkowski space with signature $(d-1,1)$ is described by the Lagrangian action

$$S=\frac{1}{2}\int d\tau\lambda^{-1}\dot{x}^2 \tag{2.1}$$

where $x^\mu(\tau)$ are the position coordinates, $\lambda(\tau)$ is an auxiliary variable and a dot denotes derivatives with respect to the parameter τ. Action (2.1) is invariant under the local infinitesimal reparametrizations

$$\delta x_\mu=\epsilon(\tau)\dot{x}_\mu, \qquad \delta\lambda=\frac{d}{d\tau}[\epsilon(\tau)\lambda] \tag{2.2}$$

and therefore describes gravity on the world-line. Action (2.1) is also invariant under the following rigid infinitesimal transformations. Poincaré transformations

$$\delta x^\mu=a^\mu+\omega^\mu_\nu x^\nu, \qquad \delta\lambda=0 \tag{2.3}$$

where $\omega_{\mu\nu}=-\omega_{\nu\mu}$ is a constant matrix, under the scale transformations

$$\delta x^\mu=\alpha x^\mu, \qquad \delta\lambda=2\alpha\lambda \tag{2.4}$$

where α is a constant, and under the conformal transformations

$$\delta x^\mu=(2x^\mu x^\nu-\eta^{\mu\nu}x^2)b_\nu, \qquad \delta\lambda=4\lambda x.b \tag{2.5}$$

where b_μ is a constant vector. Finite conformal transformations, given by [3]

$$\tilde{x}^\mu=\frac{x^\mu+b^\mu x^2}{1-2b.x+b^2x^2}, \tag{2.6a}$$

$$\tilde{\lambda}=\frac{\lambda}{(1-2b.x+b^2x^2)^2} \tag{2.6b}$$

are not globally defined, and to be well defined require a compactification of the d-dimensional Minkowski space by including the points at infinity. A possible compactification is the "quadric" described in [2]. In this paper we will not assume such a compactification, and therefore finite conformal transformations of the type (2.6) will not be considered as symmetries of action (2.1).

Although action (2.1) is not invariant under the finite conformal transformations (2.6), it is invariant under the finite scale transformation [3]

$$\tilde{x}^\mu=\exp\{\beta\}x^\mu, \qquad \tilde{\lambda}=\exp\{2\beta\}\lambda$$

where β is a constant parameter. But action (2.1) is not invariant under the local infinitesimal scale transformation $\delta x_\mu=\beta(\tau)x_\mu$, $\delta\lambda=2\beta(\tau)\lambda$, nor under the finite local scale transformation $\tilde{x}_\mu=\exp\{\beta(\tau)\}x_\mu$, $\tilde{\lambda}=\exp\{2\beta(\tau)\}\lambda$. As we will see below, although finite local scale transformations are not symmetries of action (2.1), they are symmetries of the corresponding canonical Hamiltonian. This will turn out to be related to the appearance, in massless particle theory, of the classical analogues of the old Snyder commutators, derived for the de Sitter space in the momentum representation.

As a consequence of the infinitesimal invariances (2.3), (2.4) and (2.5) of action (2.1) we can define in space-time the following vector field

$$V = a^\mu P_\mu - \frac{1}{2}\omega^{\mu\nu} M_{\mu\nu} + \alpha D + b^\mu K_\mu \tag{2.7}$$

with generators

$$P_\mu = p_\mu, \tag{2.8a}$$

$$M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \tag{2.8b}$$

$$D = x.p, \tag{2.8c}$$

$$K_\mu = 2x_\mu x.p - x^2 p_\mu \tag{2.8d}$$

P_μ generates translations in space-time, $M_{\mu\nu}$ is the generator of Lorentz transformations, D is the generator of space-time dilatations and K_μ generates conformal transformations. These generators define the algebra

$$\begin{aligned}
&\{M_{\mu\nu}, M_{\lambda\rho}\} = \eta_{\nu\lambda} M_{\mu\rho} + \eta_{\mu\rho} M_{\nu\lambda} - \eta_{\nu\rho} M_{\mu\lambda} - \eta_{\mu\lambda} M_{\nu\rho},\\
&\{M_{\mu\nu}, P_\lambda\} = \eta_{\mu\lambda} P_\nu - \eta_{\nu\lambda} P_\mu, \qquad \{M_{\mu\nu}, K_\lambda\} = \eta_{\nu\lambda} K_\mu - \eta_{\lambda\mu} K_\nu,\\
&\{D, P_\mu\} = P_\mu, \qquad \{D, K_\mu\} = -K_\mu, \qquad \{D, D\} = 0,\\
&\{K_\mu, P_\nu\} = 2(\eta_{\mu\nu} D + M_{\mu\nu}),\\
&\{D, M_{\mu\nu}\} = \{P_\mu, P_\nu\} = \{K_\mu, K_\nu\} = 0
\end{aligned} \tag{2.9}$$

computed in terms of the Poisson brackets

$$\{p_\mu, p_\nu\} = \{x_\mu, x_\nu\} = 0, \qquad \{x_\mu, p_\nu\} = \eta_{\mu\nu}. \tag{2.10}$$

The algebra (2.9) is the conformal space-time algebra. The scalar massless particle theory defined by action (2.1) is a conformal theory in d dimensions.

Conformal invariance in d dimensions is isomorphic to Lorentz invariance in $d+2$ dimensions. By defining [3]

$$L_{\mu\nu} = M_{\mu\nu}, \tag{2.11a}$$

$$L_{\mu d} = \frac{1}{2}(P_\mu + K_\mu), \tag{2.11b}$$

$$L_{\mu(d+1)} = \frac{1}{2}(P_\mu - K_\mu), \tag{2.11c}$$

$$L_{d(d+1)} = D \tag{2.11d}$$

the conformal algebra (2.9) can be put in the standard form

$$\{L_{MN}, L_{RS}\} = \eta_{MR} L_{NS} + \eta_{NS} L_{MR} - \eta_{MS} L_{NR} - \eta_{NR} L_{MS} \tag{2.12}$$

with $M, N = 0, 1, \ldots, d, d+1$ and $\eta_{MN} = \mathrm{diag}(-1, +1, \ldots, +1, -1)$. This shows that there are hidden dimensions in scalar massless particle theory. In the next section we will use these hidden dimensions to generalize the world-line action (2.1) to a more symmetric theory in a $(d+2)$-dimensional space-time.

Lagrangian mechanics is contained in Hamiltonian mechanics [14]. To be more general we must pass to the Hamiltonian formalism. In the transition to this formalism action (2.1) gives the canonical momenta

$$p_\lambda = 0, \tag{2.13}$$

$$p_\mu = \frac{\dot{x}_\mu}{\lambda} \tag{2.14}$$

and the canonical Hamiltonian

$$H = \frac{1}{2}\lambda p^2. \tag{2.15}$$

Equation (2.13) is a primary constraint [15]. Introducing the Lagrange multiplier $\xi(\tau)$ for this constraint we can write the Dirac Hamiltonian

$$H_D = \frac{1}{2}\lambda p^2 + \xi p_\lambda. \tag{2.16}$$

Requiring the dynamic stability of constraint (2.13), $\dot{p}_\lambda = \{p_\lambda, H_D\} = 0$, we obtain the secondary constraint

$$\phi = \frac{1}{2}p^2 \approx 0. \tag{2.17}$$

Constraints (2.13) and (2.17) have vanishing Poisson bracket, being therefore first-class constraints [15]. The gauge transformations generated by ϕ are discussed below. Constraint (2.13) generates translations in the arbitrary variable $\lambda(\tau)$ and can be dropped from the formalism.

In equation (2.17) we introduced [16] the *weak equality symbol* $\approx$. This is to emphasize that constraint ϕ is numerically restricted to be zero in the subspace of phase space where the canonical momentum satisfies (2.17), but ϕ does not identically vanish throughout phase space. In particular, it has nonzero Poisson brackets with the canonical positions. More generally, two functions F and G that coincide on the submanifold of phase space defined by the constraints are said to be *weakly equal* over phase space and one writes $F \approx G$. On the other hand, an equation that holds throughout phase space, and not just on the submanifold defined by the constraint equations, is called *strong*, and the usual equality symbol is used in that case. It can be demonstrated that, in general [16]

$$F \approx G \quad \Leftrightarrow \quad F - G = c_i(x, p)\phi_i \tag{2.18}$$

where ϕ_i denote the constraints.

Equation (2.17) can be treated as a constraint only if the points with $p_0 = p_1 = \cdots = p_{d-1} = 0$, corresponding to the trivial representation of the Poincaré group, are excluded from phase space [3]. From the definition of the canonical momentum (2.14) the points with $x_0 = x_1 = \cdots = x_{d-1} = 0$ must also be excluded for consistency. This introduces a non-trivial phase space topology and makes a scalar massless relativistic particle similar to the non-relativistic charge-monopole system [3, 17]. Due to this non trivial phase space topology, a flat $U(1)$ bundle will necessarily be present in the quantized massless particle theory.

To further develop the Hamiltonian formalism, we write action (2.1) in the form

$$S=\int_{\tau_i}^{\tau_f} d\tau\left(\dot{x}.p-\frac{1}{2}\lambda p^2\right). \tag{2.19}$$

If we solve the equation of motion for p_μ that follows from (2.19) and insert the result back in it, we recover action (2.1). Constraint (2.17) generates the local infinitesimal transformation

$$\delta x_\mu=\epsilon(\tau)\{x_\mu,\phi\}=\epsilon(\tau)p_\mu, \tag{2.20a}$$

$$\delta p_\mu=\epsilon(\tau)\{p_\mu,\phi\}=0, \tag{2.20b}$$

$$\delta\lambda=\dot{\epsilon}(\tau) \tag{2.20c}$$

under which action (2.19) transforms as

$$\delta S=\int_{\tau_i}^{\tau_f} d\tau\frac{d}{d\tau}(\epsilon\phi). \tag{2.21}$$

Since the interval (τ_i,τ_f) is arbitrary, we see that action (2.19) is invariant under transformations (2.20), and that the quantity $Q=\epsilon\phi$ can be interpreted as the conserved Hamiltonian Noether charge or as the generator of the local transformations (2.20), depending on wether the equations of motion are satisfied or not [18]. This particular aspect of the quantity Q will be used as a consistency check when we introduce vector fields in 2T physics below.

The most general physically permissible motion should allow for an arbitrary gauge transformation to be performed while the system is dynamically evolving in time [16]. Since the dynamic time evolution of a physical system is governed by its Hamiltonian, this arbitrary gauge transformation must leave the Hamiltonian invariant. In the case of the scalar relativistic massless particle, parametrized by τ, we point out that the Hamiltonian (2.15) is invariant under the finite local scale transformations

$$\tilde{p}_\mu=\exp\{-\beta(\tau)\}p_\mu, \tag{2.22a}$$

$$\tilde{\lambda}=\exp\{2\beta(\tau)\}\lambda \tag{2.22b}$$

where $\beta(\tau)$ is an arbitrary scalar function. From (2.14) for the canonical momentum we find that x^μ transforms as

$$\tilde{x}^\mu=\exp\{\beta(\tau)\}x^\mu \tag{2.22c}$$

when p_μ transforms as in (2.22a). The finite local scale transformation (2.22) is a symmetry in phase space but, as we saw above, it breaks down if we try a transition to configuration space. It is interesting, in the case when $\beta(\tau)=\beta(x(\tau))$, to try to relate, using the correspondence principle, the local scale transformation (2.22c) of the position variables with the local phase transformation (1.1) of the position eigenstates. Gravity and the flat $U(1)$ bundle would then be related by finite local scale invariance. We will not consider this point here.

Consider now the bracket structure that transformations (2.22a) and (2.22c) induce in phase space. The following calculations are an improved, more rigorous version, of the ones which appear in [20]. Retaining only the linear terms in β in the exponentials, we find that the new transformed canonical variables $(\tilde{x}_\mu,\tilde{p}_\mu)$ obey the brackets

$$\{\tilde{p}_\mu,\tilde{p}_\nu\}=(\beta-1)[\{p_\mu,\beta\}p_\nu+p_\mu\{\beta,p_\nu\}]+\{\beta,\beta\}p_\mu p_\nu, \tag{2.23a}$$

$$\{\tilde{x}_\mu, \tilde{p}_\nu\} = (1+\beta)[\delta_{\mu\nu}(1-\beta) - \{x_\mu, \beta\}p_\nu] + (1-\beta)x_\mu\{\beta, p_\nu\} - \{\beta,\beta\}x_\mu p_\nu, \tag{2.23b}$$

$$\{\tilde{x}_\mu, \tilde{x}_\nu\} = (1+\beta)[x_\mu\{\beta, x_\nu\} - x_\nu\{\beta, x_\mu\}] + \{\beta,\beta\}x_\mu x_\nu. \tag{2.23c}$$

If we choose $\beta = \phi$ in (2.23) and compute the brackets on the right side in terms of the Poisson brackets (2.10), we find the expressions, after dropping terms proportional to $\beta^2 = \phi^2$

$$\{\tilde{p}_\mu, \tilde{p}_\nu\} = 0, \tag{2.24a}$$

$$\{\tilde{x}_\mu, \tilde{p}_\nu\} = \eta_{\mu\nu} - p_\mu p_\nu, \tag{2.24b}$$

$$\{\tilde{x}_\mu, \tilde{x}_\nu\} = -M_{\mu\nu} - M_{\mu\nu}\phi. \tag{2.24c}$$

Now, keeping the same order of approximation used to arrive at brackets (2.23), that is, retaining only the linear terms in β, the transformation equations (2.22a) and (2.22c) read

$$\tilde{p}_\mu = \exp\{-\beta\}p_\mu = (1-\beta)p_\mu, \tag{2.25a}$$

$$\tilde{x}_\mu = \exp\{\beta\}x_\mu = (1+\beta)x_\mu. \tag{2.25b}$$

Using again the same function $\beta = \phi$ in (2.25), we write them as

$$\tilde{p}_\mu - p_\mu = c_\mu(x,p)\phi, \tag{2.26a}$$

$$\tilde{x}_\mu - x_\mu = d_\mu(x,p)\phi \tag{2.26b}$$

where $c_\mu(x,p) = -p_\mu$ and $d_\mu(x,p) = x_\mu$. Equations (2.26) are in the form (2.18) and so we can write

$$\tilde{p}_\mu \approx p_\mu, \qquad \tilde{x}_\mu \approx x_\mu. \tag{2.27}$$

Using (2.18) and (2.27) in brackets (2.24), we can finally write the phase space brackets

$$\{p_\mu, p_\nu\} \approx 0, \tag{2.28a}$$

$$\{x_\mu, p_\nu\} \approx \eta_{\mu\nu} - p_\mu p_\nu, \tag{2.28b}$$

$$\{x_\mu, x_\nu\} \approx -M_{\mu\nu}. \tag{2.28c}$$

In a transition to the quantum theory by the correspondence principle rule that [commutators] $=i\hbar$\{brackets\}, the brackets (2.28) will reproduce the Snyder commutators [19] in the case when the noncommutativity parameter is $\theta = 1$. The Snyder commutators were obtained in a projective geometry approach to the de Sitter space in the momentum representation. Here we have derived their classical correspondents from the finite local scale invariance (2.22) of the scalar massless particle Hamiltonian. However, we have not succeeded in obtaining from the finite local scale invariance (2.22) the Riemannian metric structure required by quantum dynamics in the position representation. The massless particle does not have enough gauge freedom for this metric structure to be derived in the same way we derived the momentum space brackets (2.28). This is because the canonical Hamiltonian (2.15) explicitly distinguishes momentum from position. As we will see in the next section, this situation changes in 2T physics, where momentum and position are indistinguishable variables, and a metric structure can be derived in $d+2$ dimensions in exactly the same way we derived the d dimensional momentum space brackets (2.28).

It can be verified that brackets (2.28) satisfy all Jacobi identities among the canonical variables, preserve the d dimensional conformal algebra (2.9) and preserve the first class property of constraint (2.17), therefore preserving gauge invariance. Due to the non-trivial topology of the massless particle configuration space, a vector field of vanishing strength tensor must be present in the quantum theory. We will not consider this point here. Instead we will concentrate on a discussion of this same situation in classical 2T physics, which contains the d dimensional massless scalar relativistic particle as a gauge-fixed subsystem. For a general parametrization of the classical solutions of 2T physics in any gauge, see [7].

Hamiltonian (2.15) gives the classical equations of motion

$$\dot{x}_\mu = \{x_\mu, H\} = \lambda p_\mu, \tag{2.29a}$$

$$\dot{p}_\mu = \{p_\mu, H\} = 0. \tag{2.29b}$$

Equation (2.29b) shows that the massless particle moves with a constant momentum relative to the parameter τ, and is therefore a freely moving particle. This situation changes in 2T physics because the $Sp(2, R)$ local invariance or, in other words, the local indistinguishability between position and momentum, brings with it an intrinsic interaction and as a result a massless relativistic particle in a $d + 2$ dimensional space-time can no longer be completely free. The idea in this paper is that it feels the effect of an intrinsic curved $d + 2$ dimensional background.

3 Two-Time Physics

In the usual one-time (1T) physics, a metric structure appears in the most general configuration space formulation of quantum mechanics [1]. 1T physics has been, and can always be used, to confirm the predictions of 2T physics. In this paper we follow the opposite route. This route is to investigate the possible existence of the $d + 2$ dimensional generalization of a well known situation in 1T physics. Specifically, here we are interested in the construction of a $d + 2$ dimensional general formulation of quantum mechanics. This general formulation is expected to contain non relativistic quantum mechanics in $d - 1$ dimensions and relativistic quantum mechanics in d dimensions as gauge-fixed subsectors. However, before trying to construct such a theory, we must verify if its basic ingredients are available. In this section we show how a natural metric structure can be found in $d + 2$ dimensions. We start by reviewing the basic ideas that led to 2T physics.

The quantization rules of quantum mechanics are symmetric under the interchange of coordinates and momenta. This is known as the discrete symplectic symmetry $Sp(2)$ that transforms (x, p) as a doublet. The central idea in two-time physics [6–12] is to introduce a new gauge invariance in phase space by gauging the duality of the quantum commutator $[X_M, P_N] = i\hbar\eta_{MN}$. This procedure leads to a symplectic $Sp(2, R)$ gauge theory. To remove the distinction between position and momentum we set $X_1^M = X^M$ and $X_2^M = P^M$ and define the doublet $X_i^M = (X_1^M, X_2^M)$. The local $Sp(2, R)$ acts as

$$\delta X_i^M(\tau) = \epsilon_{ik}\omega^{kl}(\tau)X_l^M(\tau) \tag{3.1}$$

$\omega^{ij}(\tau)$ is a symmetric matrix containing three local parameters and ϵ_{ij} is the Levi-Civita symbol that serves to raise or lower indices. The $Sp(2, R)$ gauge field A^{ij} is symmetric in (i, j) and transforms as

$$\delta A^{ij} = \partial_\tau\omega^{ij} + \omega^{ik}\epsilon_{kl}A^{lj} + \omega^{jk}\epsilon_{kl}A^{il}. \tag{3.2}$$

The covariant derivative is

$$D_\tau X_i^M = \partial_\tau X_i^M - \epsilon_{ik} A^{kl} X_l^M. \tag{3.3}$$

An action invariant under the $Sp(2, R)$ gauge symmetry is

$$S = \frac{1}{2}\int d\tau (D_\tau X_i^M)\epsilon^{ij} X_j^N \eta_{MN}. \tag{3.4}$$

In Hamiltonian form action (3.4) becomes

$$S = \int d\tau \left[\dot{X}.P - \left(\frac{1}{2}\lambda_1 P^2 + \lambda_2 X.P + \frac{1}{2}\lambda_3 X^2\right)\right] \tag{3.5}$$

where λ_α, $\alpha = 1, 2, 3$ are Lagrange multipliers and the canonical Hamiltonian is

$$H = \frac{1}{2}\lambda_1 P^2 + \lambda_2 X.P + \frac{1}{2}\lambda_3 X^2. \tag{3.6}$$

The equations of motion for the λ's give the first-class constraints

$$\phi_1 = \frac{1}{2}P^2 \approx 0, \tag{3.7}$$

$$\phi_2 = X.P \approx 0, \tag{3.8}$$

$$\phi_3 = \frac{1}{2}X^2 \approx 0. \tag{3.9}$$

Constraints (3.7–3.9), as well as evidences of two-time physics, were independently obtained in [3]. The presence of first class constraints and the associated gauge freedom indicates that there is more than one set of canonical variables that corresponds to a given physical state [16]. However, equations (3.7) and (3.9) can be treated as constraints only if the hypersurfaces $X_0 = X_1 = \cdots = X_{d+1} = 0$ and $P_0 = P_1 = \cdots = P_{d+1} = 0$ are excluded from phase space. Only in this case the gauge orbits generated by ϕ_1 and ϕ_3 are regular [3, 16]. Here then we also have a phase space with a non-trivial topology. If we consider the Euclidean, or the Minkowski metric as the background space-time, we find that the surface defined by the constraint equations (3.7–3.9) is trivial. The only metric giving a non-trivial surface, preserving the unitarity of the theory, and avoiding the ghost problem is a flat metric with two time-like dimensions [6–12]. Following [6–12] we introduce another space-like dimension and another time-like dimension and start working in a Minkowski space with signature $(d, 2)$. Action (3.5) is the $d + 2$ dimensional generalization of the d dimensional massless particle action (2.19). Action (3.5) describes conformal gravity on the world-line [7, 23, 24]. Constraints (3.7–3.9) can also be interpreted as describing a massless particle living on the border of a $d + 1$ dimensional AdS space of infinite radius [3].

In terms of the Poisson brackets

$$\{P_M, P_N\} = \{X_M, X_N\} = 0, \qquad \{X_M, P_N\} = \eta_{MN} \tag{3.10}$$

the local infinitesimal $Sp(2, R)$ transformations of action (3.5) are

$$\delta X_M = \epsilon_\alpha(\tau)\{X_M, \phi_\alpha\} = \epsilon_1 P_M + \epsilon_2 X_M, \tag{3.11a}$$

$$\delta P_M = \epsilon_\alpha(\tau)\{P_M, \phi_\alpha\} = -\epsilon_2 P_M - \epsilon_3 X_M, \tag{3.11b}$$

$$\delta\lambda_1 = \dot{\epsilon}_1 + 2\epsilon_2\lambda_1 - 2\epsilon_1\lambda_2, \tag{3.11c}$$

$$\delta\lambda_2 = \dot{\epsilon}_2 + \epsilon_3\lambda_1 - \epsilon_1\lambda_3, \tag{3.11d}$$

$$\delta\lambda_3 = \dot{\epsilon}_3 + 2\epsilon_3\lambda_2 - 2\epsilon_2\lambda_3, \tag{3.11e}$$

under which

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \frac{d}{d\tau}(\epsilon_\alpha \phi_\alpha). \tag{3.12}$$

As in the massless particle case, since the interval (τ_i, τ_f) is arbitrary, the quantity $Q = \epsilon_\alpha \phi_\alpha$ with $\alpha = 1, 2, 3$ can be interpreted as the conserved Hamiltonian Noether charge, or as the generator of the local infinitesimal transformations (3.11), depending on wether the equations of motion are satisfied or not [18].

Rigid infinitesimal $SO(d, 2)$ transformations have the generator [6–12]

$$L_{MN} = X_M P_N - X_N P_M. \tag{3.13}$$

The L_{MN} satisfy the algebra (2.12) and generate the transformations

$$\delta X_M = -\frac{1}{2}\omega_{RS}\{X_M, L_{RS}\} = \omega_{MR} X_R, \tag{3.14a}$$

$$\delta P_M = -\frac{1}{2}\omega_{RS}\{P_M, L_{RS}\} = \omega_{MR} P_R, \tag{3.14b}$$

$$\delta\lambda_\alpha = 0 \tag{3.14c}$$

under which $\delta S = 0$. Because the L_{MN} are gauge invariant, $\{L_{MN}, \phi_\alpha\} = 0$, the $SO(d, 2)$ invariance is also present in all the d dimensional relativistic systems that can be obtained from the 2T physics action (3.5) by imposing two gauge conditions, and in all the $(d-1)$ dimensional non-relativistic systems that can be obtained from (3.5) by imposing three gauge conditions.

Let us now consider how a Riemannian metric structure can be induced in the $d + 2$ dimensional flat space-time of 2T physics. The 2T Hamiltonian (3.6) is invariant under the finite local scale transformations

$$\tilde{X}^M = \exp\{\beta(\tau)\} X^M, \tag{3.15a}$$

$$\tilde{P}_M = \exp\{-\beta(\tau)\} P_M, \tag{3.15b}$$

$$\tilde{\lambda}_1 = \exp\{2\beta(\tau)\}\lambda_1, \tag{3.15c}$$

$$\hat{\lambda}_2 = \lambda_2, \tag{3.15d}$$

$$\tilde{\lambda}_3 = \exp\{-2\beta(\tau)\}\lambda_3 \tag{3.15e}$$

where $\beta(\tau)$ is an arbitrary scalar function. The subsequent steps are simply higher dimensional extensions of those for the massless particle. Keeping only the linear terms in β in transformation (3.15), we arrive at the brackets

$$\{\tilde{P}_M, \tilde{P}_N\} = (\beta - 1)[\{P_M, \beta\} P_N + \{\beta, P_N\} P_M] + \{\beta, \beta\} P_M P_N \tag{3.16a}$$

$$\begin{aligned}\{\tilde{X}_M, \tilde{P}_N\} = {} & (1 + \beta)[\eta_{MN}(1 - \beta) - \{X_M, \beta\} P_N] \\ & + (1 - \beta) X_M \{\beta, P_N\} - X_M X_N \{\beta, \beta\}\end{aligned} \tag{3.16b}$$

$$\{\tilde{X}_M, \tilde{X}_N\} = (1+\beta)[X_M\{\beta, X_N\} - X_N\{\beta, X_M\}] + X_M X_N\{\beta, \beta\}. \tag{3.16c}$$

If we choose $\beta = \phi_1$ in (3.16) and compute the brackets on the right side using the Poisson brackets (3.10), we find the expressions, after dropping terms proportional to $\beta^2 = \phi_1^2$

$$\{\tilde{P}_M, \tilde{P}_N\} = 0, \tag{3.17a}$$

$$\{\tilde{X}_M, \tilde{P}_N\} = \eta_{MN} - P_M P_N, \tag{3.17b}$$

$$\{\tilde{X}_M, \tilde{X}_N\} = -L_{MN} - L_{MN}\phi_1. \tag{3.17c}$$

In the same order of approximation used to arrive at brackets (3.16), transformation equations (3.15a) and (3.15b) read

$$\tilde{X}^M = \exp\{\beta\} X^M = (1+\beta) X^M, \tag{3.18a}$$

$$\tilde{P}_M = \exp\{-\beta\} P_M = (1-\beta) P_M. \tag{3.18b}$$

Using again the same function $\beta = \phi_1$ in (3.18), we write them as

$$\tilde{X}^M - X^M = C^M_\alpha(X, P)\phi_\alpha, \tag{3.19a}$$

$$\tilde{P}_M - P_M = D^\alpha_M(X, P)\phi_\alpha \tag{3.19b}$$

with $C^M_1 = X^M$, $C^M_2 = C^M_3 = 0$ and $D^1_M = -P_M$, $D^2_M = D^3_M = 0$. Equations (3.19) are in the form (2.18) and so we can write

$$\tilde{X}^M \approx X^M, \qquad \tilde{P}_M \approx P_M. \tag{3.20}$$

Using these weak equalities in brackets (3.17) we can write the phase space brackets

$$\{P_M, P_N\} \approx 0, \tag{3.21a}$$

$$\{X_M, P_N\} \approx \eta_{MN} - P_M P_N, \tag{3.21b}$$

$$\{X_M, X_N\} \approx -L_{MN}. \tag{3.21c}$$

Brackets (3.21) are the $(d+2)$ dimensional extensions of the d dimensional momentum space brackets (2.28) we found for the massless relativistic particle. But in 2T physics, where X_M and P_M are locally indistinguishable variables, brackets (3.21) have a dual version in position space. We can perform the duality transformation

$$X_M \to P_M, \tag{3.22a}$$

$$P_M \to -X_M, \tag{3.22b}$$

$$\lambda_1 \to \lambda_3, \qquad \lambda_2 \to -\lambda_2, \qquad \lambda_3 \to \lambda_1 \tag{3.22c}$$

which leaves the 2T Hamiltonian (3.6) invariant, and under which the 2T action (3.5) transforms as $\delta S = -\int_{\tau_i}^{\tau_f} d\tau \frac{d}{d\tau}(X.P)$, being therefore invariant up to a surface term. However, we can not simply substitute the duality transformations (3.22a) and (3.22b) in bracket (3.21b) in order to obtain a metric structure in position space in $d+2$ dimensions. This procedure introduces incorrect minus signs in some of the resultant brackets and as a result the Jacobi identities involving position and momentum fail to close. This is because, as we saw in the introduction, the gravitational field, regarded as a gauge field, corresponds to the group of continuous local scale transformations and not to duality transformations of the type (3.22). The correct procedure starts by noting that transformation (3.22) changes the function $\beta = \frac{1}{2}P^2$, we used to arrive at brackets (3.21) into the new function $\beta = \frac{1}{2}X^2$. Introducing this new function into brackets (3.16), which are the consequences in phase space of the presence of the finite local scale invariance (3.15) of the 2T Hamiltonian, and performing the same steps as in the case $\beta = \frac{1}{2}P^2$, we arrive at the position space brackets

$$\{P_M, P_N\} \approx L_{MN}, \tag{3.23a}$$

$$\{X_M, P_N\} \approx \eta_{MN} + X_M X_N, \tag{3.23b}$$

$$\{X_M, X_N\} \approx 0. \tag{3.23c}$$

Notice that we can not obtain brackets (3.23) by performing the duality transformation (3.22) in brackets (3.21). From equation (3.23b) we see that we can use the finite local scale invariance (3.15) of the 2T Hamiltonian to change from the flat Minkowski space with metric η_{MN} to a Riemannian space with metric tensor

$$G_{MN} = \eta_{MN} + X_M X_N. \tag{3.24}$$

This procedure of incorporating gravitational effects into quantum mechanics by modifying the commutator $[X_M, P_N]$ (or the corresponding classical bracket, as is the case here) is not new and in the usual 1T physics it becomes unavoidable [25] at energy scales near the Planck scale. In 2T physics this procedure can not change the dynamic evolution of the system because the Hamiltonian (3.6) is invariant under the local scale transformation (3.15). In fact, Hamiltonian (3.6) generates the classical equations of motion

$$\dot{X}_M = \{X_M, H\} = \lambda_1 P_M + \lambda_2 X_M, \tag{3.25a}$$

$$\dot{P}_M = \{P_M, H\} = -\lambda_2 P_M - \lambda_3 X_M \tag{3.25b}$$

computed in terms of the Poisson brackets (3.10). Equation (3.25b) shows that the particle's momentum is no longer constant relative to the parameter τ. An interaction is perceived by the massless particle as a result of its embedding in $d+2$ dimensions. The idea here is that it feels the effect of the background (3.24).

It is easy to verify that if we leave the $d+2$ dimensional Minkowski space of 2T physics, and use the local scale transformation (3.15) to change to the $d+2$ dimensional space with metric tensor (3.24), the new Hamiltonian will differ from (3.6) by terms that are quadratic in the first class constraints. These quadratic terms can be dropped, and in the linear approximation the Hamiltonian in the background (3.24) is identical to (3.6). In addition, the equations of motion computed using the Hamiltonian (3.6) and brackets (3.23) differ from the equations of motion (3.25) by terms that are linear in the constraints. These linear terms can also be dropped and the equations of motion in the background (3.24), computed in terms of brackets (3.23), are identical to (3.25). We are then forced to conclude that the $d+2$ dimensional space with metric tensor (3.24) is an equally valid natural background for 2T physics because no homothety condition [12] is necessary here. This was unknown until now. More surprising is that, after dropping the terms proportional to the constraints, the Hamiltonian and the equations of motion in the momentum space background

$$\bar{G}_{MN} = \eta_{MN} - P_M P_N \tag{3.26}$$

where brackets (3.21) are valid, are also identical to (3.6) and (3.25), respectively. The flat $d+2$ dimensional Minkowski space is not the only possible space for 2T physics. The $d+2$ dimensional position space (3.24) and the $d+2$ dimensional momentum space (3.26) are also possible and more general spaces for 2T physics. The important point here is that, reasoning in analogy to known results in 1T physics [1], we may expect that the transition to these more general $d+2$ dimensional backgrounds will be necessary to guarantee the correct normalization of the position and momentum eigenstates, the correct spectral decomposition of the identity operator in the position and momentum eigenbasis, the correct matrix elements of the position and momentum operators, and also the correct integration measure for the inner product of any two states in a general configuration space or momentum space formulation of quantum mechanics in $d+2$ dimensions.

To conclude this section we mention that in our derivation of brackets (3.21) and (3.23) there is no need to use the Dirac bracket because there is no second class constraint to begin with. Dirac brackets would have appeared if we had imposed gauge conditions to turn the first class constraints (3.7–3.9) into second class ones. This would bring us back to $d-1$ dimensions. An example of this is that the d dimensional brackets (2.28), we obtained for the massless particle using scale invariance arguments, can also be derived as a Dirac bracket after imposing two canonical gauge conditions [30] which turn the first class constraints (3.8) and (3.9) of 2T physics into second class constraints. In this paper, this restriction of the gauge freedom using the Dirac bracket technique, although possible, is not necessary. This will guarantee that we are in $d+2$ dimensions. Bypassing the Dirac brackets is a substantial advantage for our purposes here. Indeed, the quantum realization of Dirac brackets that depend on the canonical variables may be highly nontrivial and is by no means guaranteed [16].

4 2T Physics with Topological Vector Fields

Now we explicitly take into account the non-trivial phase space topology of 2T physics. This can be done by introducing a vector field $A_M(X)$ which defines a section of a flat $U(1)$ bundle over space-time [1]. The vector field must have a vanishing antisymmetric second rank strength tensor, $F_{MN}=\partial_M A_N-\partial_N A_M=0$. When this condition is met, the flat $U(1)$ bundle may be characterized [1], up to local infinitesimal reparametrizations, by the differential 1-form $dX^M A_M(X)$.

As we saw in section three, to obtain regular gauge orbits for the first class constraints of 2T physics, the origin of phase space must be removed. This creates a topological obstruction to the reduction of the vector field $A_M(X)$ to a pure gauge, $A_M=\frac{\partial\chi(X)}{\partial X^M}$, where $\chi(X)$ is an arbitrary function. In other words, the vector field must be present in the quantized 2T theory. In this section we search for a corresponding classical action. As an initial attempt we modify action (3.5) according to the usual minimal coupling prescription to vector fields, $P_M\to P_M-A_M$. This produces the correct $U(1)$ covariant derivative in the quantum theory [1]. The 2T action in this case is

$$S=\int d\tau\left\{\dot{X}.P-\left[\frac{1}{2}\lambda_1(P-A)^2+\lambda_2 X.(P-A)+\frac{1}{2}\lambda_3 X^2\right]\right\} \tag{4.1}$$

where the Hamiltonian is

$$H=\frac{1}{2}\lambda_1(P-A)^2+\lambda_2 X.(P-A)+\frac{1}{2}\lambda_3 X^2. \tag{4.2}$$

The equations of motion for the multipliers now give the constraints

$$\phi_1 = \frac{1}{2}(P - A)^2 \approx 0, \tag{4.3}$$

$$\phi_2 = X.(P - A) \approx 0, \tag{4.4}$$

$$\phi_3 = \frac{1}{2}X^2 \approx 0. \tag{4.5}$$

The Poisson brackets between the canonical variables and the vector field are

$$\{X_M, A_N\} = 0, \tag{4.6a}$$

$$\{P_M, A_N\} = -\frac{\partial A_N}{\partial X^M}, \tag{4.6b}$$

$$\{A_M, A_N\} = 0. \tag{4.6c}$$

Computing the algebra of constraints (4.3–4.5) using the Poisson brackets (3.10) and (4.6) we obtain the equations

$$\{\phi_1, \phi_1\} = (P^M - A^M)F_{MN}(P^N - A^N), \tag{4.7a}$$

$$\begin{aligned}\{\phi_1, \phi_2\} = {} & -2\phi_1 + (P^M - A^M)\frac{\partial}{\partial X^M}(X.A) - (P - A).A \\ & - X^M \frac{\partial}{\partial X^M}[(P - A).A] - X^M \frac{\partial}{\partial X^M}\left(\frac{1}{2}A^2\right),\end{aligned} \tag{4.7b}$$

$$\{\phi_2, \phi_2\} = X^M F_{MN} X^N, \tag{4.7c}$$

$$\{\phi_1, \phi_3\} = -\phi_2, \tag{4.7d}$$

$$\{\phi_2, \phi_3\} = -2\phi_3, \tag{4.7e}$$

$$\{\phi_3, \phi_3\} = 0. \tag{4.7f}$$

For the case in which we are interested in this paper, we see from the above equations that constraints (4.3–4.5) become first class constraints when the vector field satisfies the conditions

$$F_{MN} = 0, \tag{4.8a}$$

$$X.A = 0, \tag{4.8b}$$

$$(P - A).A = 0, \tag{4.8c}$$

$$\frac{1}{2}A^2 = 0. \tag{4.8d}$$

Condition (4.8a) implies that the vector field $A_M(X)$ defines a section of a flat $U(1)$ bundle over the $d+2$ dimensional space-time. Observe that in the case when $F_{MN} \neq 0$ the vanishing of bracket (4.7c) leads to the same condition (1.4) obtained in [12]. But here a careful look at bracket (4.7a) suggests that, in the presence of a vector field for which $F_{MN} \neq 0$, condition (1.4) should be complemented with the condition $(P^M - A^M)F_{MN} = 0$. This would render the theory simultaneously in agreement with the minimal coupling prescription to vector fields and with the local indistinguishability between X^M and $P^M - A^M(X)$ in the presence of the vector field. A curious observation is that P^M also becomes indistinguishable from $X^M - A^M(P)$. This point will be considered in a future paper [22].

As can be easily verified, conditions (4.8b–4.8d) imply that constraints (4.3–4.5) are not the irreducible [16] set of constraints for 2T physics with a topological vector field. Combining then conditions (4.8b–4.8d) with constraints (4.3–4.5), we obtain the irreducible set of constraints

$$\phi_1 = \frac{1}{2}P^2 \approx 0, \tag{4.9}$$

$$\phi_2 = X.P \approx 0, \tag{4.10}$$

$$\phi_3 = \frac{1}{2}X^2 \approx 0, \tag{4.11}$$

$$\phi_4 = X.A \approx 0, \tag{4.12}$$

$$\phi_5 = P.A \approx 0, \tag{4.13}$$

$$\phi_6 = \frac{1}{2}A^2 \approx 0. \tag{4.14}$$

Observe that Dirac's conditions (1.2a) and (1.2b) are now reproduced by constraints ϕ_4 and ϕ_5. The contrast with the set (1.2) is that our calculation leads to a scalar third condition on the vector field, a condition which will now be verified to be the correct constraint for the 2T theory in the presence of a vector field for which $F_{MN} = 0$.

It can be verified that constraints (4.9–4.14) are all first class. We can then write down the Hamiltonian action

$$S = \int_{\tau_i}^{\tau_f} d\tau \left[\dot{X}.P - \left(\frac{1}{2}\lambda_1 P^2 + \lambda_2 X.P + \frac{1}{2}\lambda_3 X^2 + \lambda_4 X.A + \lambda_5 P.A + \frac{1}{2}\lambda_6 A^2 \right) \right] \tag{4.15}$$

describing two-time physics with a vector field of topological origin. The Hamiltonian is

$$H = \frac{1}{2}\lambda_1 P^2 + \lambda_2 X.P + \frac{1}{2}\lambda_3 X^2 + \lambda_4 X.A + \lambda_5 P.A + \frac{1}{2}\lambda_6 A^2. \tag{4.16}$$

The L_{MN} in (3.13) generate the rigid infinitesimal $SO(d,2)$ transformations in action (4.15)

$$\delta X_M = -\frac{1}{2}\omega_{RS}\{X_M, L_{RS}\} = \omega_{MR} X_R, \tag{4.17a}$$

$$\delta P_M = -\frac{1}{2}\omega_{RS}\{P_M, L_{RS}\} = \omega_{MR} P_R, \tag{4.17b}$$

$$\delta A_M = \frac{\partial A_M}{\partial X_R}\delta X_R, \tag{4.17c}$$

$$\delta\lambda_\varrho = 0, \quad \varrho = 1, 2, \ldots, 6 \tag{4.17d}$$

under which $\delta S = 0$. It can be checked that L_{MN} has weakly vanishing brackets with the first class constraints (4.9–4.14), being therefore gauge invariant.

Action (4.15) also has the local infinitesimal invariance

$$\delta X_M = \epsilon_\varrho(\tau)\{X_M, \phi_\varrho\} = \epsilon_1 P_M + \epsilon_2 X_M + \epsilon_5 A_M, \tag{4.18a}$$

$$\delta P_M = \epsilon_\varrho(\tau)\{P_M, \phi_\varrho\} = -\epsilon_2 P_M - \epsilon_3 X_M - \epsilon_4 A_M - \epsilon_4 X_N \frac{\partial A_N}{\partial X^M} - \epsilon_5 P_N \frac{\partial A_N}{\partial X^M} - \epsilon_6 A_N \frac{\partial A_N}{\partial X^M}, \tag{4.18b}$$

$$\delta A_M = \frac{\partial A_M}{\partial X^N}\delta X^N, \tag{4.18c}$$

$$\delta\lambda_1 = \dot{\epsilon}_1 + 2\epsilon_2\lambda_1 - 2\epsilon_1\lambda_2, \tag{4.18d}$$

$$\delta\lambda_2 = \dot{\epsilon}_2 + \epsilon_3\lambda_1 - \epsilon_1\lambda_3, \tag{4.18e}$$

$$\delta\lambda_3 = \dot{\epsilon}_3 + 2\epsilon_3\lambda_2 - 2\epsilon_2\lambda_3, \tag{4.18f}$$

$$\delta\lambda_4 = \dot{\epsilon}_4 + \epsilon_3\lambda_5 - \epsilon_5\lambda_3, \tag{4.18g}$$

$$\delta\lambda_5 = \dot{\epsilon}_5 + \epsilon_2\lambda_5 - \epsilon_5\lambda_2 \tag{4.18h}$$

$$\delta\lambda_6 = \dot{\epsilon}_6 \tag{4.18i}$$

under which

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \frac{d}{d\tau}(\epsilon_\rho\phi_\rho). \tag{4.19}$$

Now the conserved charge, or the generator of the local transformations, depending on wether the equations of motion are satisfied or not, is the quantity $Q = \epsilon_\rho\phi_\rho$ with $\rho = 1, 2, \dots, 6$. This generalizes the local infinitesimal invariance (3.11) of 2T physics to the case when a vector field of vanishing strength tensor is present.

Hamiltonian (4.16) is invariant under the finite local scale transformations

$$\tilde{X}^M = \exp\{\beta(\tau)\}X^M, \tag{4.20a}$$

$$\tilde{P}_M = \exp\{-\beta(\tau)\}P_M, \tag{4.20b}$$

$$\tilde{A}_M = \exp\{-\beta(\tau)\}A_M, \tag{4.20c}$$

$$\tilde{\lambda}_1 = \exp\{2\beta(\tau)\}\lambda_1, \tag{4.20d}$$

$$\tilde{\lambda}_2 = \lambda_2, \tag{4.20e}$$

$$\tilde{\lambda}_3 = \exp\{-2\beta(\tau)\}\lambda_3, \tag{4.20f}$$

$$\tilde{\lambda}_4 = \lambda_4, \tag{4.20g}$$

$$\tilde{\lambda}_5 = \exp\{2\beta(\tau)\}\lambda_5, \tag{4.20h}$$

$$\tilde{\lambda}_6 = \exp\{2\beta(\tau)\}\lambda_6. \tag{4.20i}$$

Notice that transformations (4.20b) and (4.20c) are consistent with the minimal coupling prescription to vector fields. Using the invariance (4.20) we can arrive, if we choose $\beta = \phi_1$, at the same brackets (3.21). We can arrive at brackets (3.23) by choosing $\beta = \phi_3$ as before. The metric structure (3.24) in position space and the metric structure (3.26) in momentum space we obtained in section three are then both preserved in the presence of a vector field of vanishing strength tensor.

In the presence of the vector field we can again change to the backgrounds (3.24) or (3.26) without changing the dynamic evolution of the system. For instance, if we perform the change to the background (3.24) we find that the new Hamiltonian differs from (4.16) by terms that are quadratic in the first class constraints (4.9–4.14). These quadratic terms can again be dropped and in the linear approximation the Hamiltonian in the background (3.24), and in the presence of the vector field, is identical to (4.16). The classical equations of motion generated by the Hamiltonian (4.16), computed in terms of the Poisson brackets (3.10) and (4.6), are

$$\dot{X}_M = \{X_M, H\} = \lambda_1 P_M + \lambda_2 X_M + \lambda_5 A_M, \tag{4.21a}$$

$$\dot{P}_M = \{P_M, H\} = -\lambda_2 P_M - \lambda_3 X_M - \lambda_4 A_M - \lambda_4 X_N \frac{\partial A_N}{\partial X^M} - \lambda_5 P_N \frac{\partial A_N}{\partial X^M} - \lambda_6 A_N \frac{\partial A_N}{\partial X^M}, \tag{4.21b}$$

$$\dot{A}_M = \{A_M, H\} = \lambda_1 P_N \frac{\partial A_M}{\partial X^N} + \lambda_2 X_N \frac{\partial A_M}{\partial X^N} + \lambda_5 A_N \frac{\partial A_M}{\partial X^N}. \tag{4.21c}$$

The local scale transformation (4.20) with the function $\beta = \frac{1}{2}X^2$ changes the Poisson brackets (4.6) into the new set

$$\{X_M, A_N\} = 0, \tag{4.22a}$$

$$\{P_M, A_N\} = -\frac{\partial A_N}{\partial X^M} + X_M A_N, \tag{4.22b}$$

$$\{A_M, A_N\} = 0. \tag{4.22c}$$

Now, computing the equations of motion generated by the Hamiltonian (4.16) in terms of the brackets (3.23) and (4.22), we find that these equations differ from equations (4.21) by terms that are linear in the first class constraints (4.9–4.14). These terms can be dropped and the classical equations of motion in the background (3.24) become identical to (4.21), which are valid in the flat $d+2$ dimensional background. The same situation occurs in the background (3.26) after the Poisson brackets (4.6) are replaced by the brackets

$$\{X_M, A_N\} = -P_M A_N - X_M P_S \frac{\partial A_N}{\partial X^S}, \tag{4.23a}$$

$$\{P_M, A_N\} = -\frac{\partial A_N}{\partial X^M} + P_M P_S \frac{\partial A_N}{\partial X^S}, \tag{4.23b}$$

$$\{A_M, A_N\} = A_M P_S \frac{\partial A_N}{\partial X^S} - A_N P_S \frac{\partial A_M}{\partial X^S} \tag{4.23c}$$

which emerge after the local scale transformation (4.20) with $\beta = \frac{1}{2}P^2$ is performed. Also in the presence of the vector field, the $d+2$ dimensional Minkowski space is not the only possible space for 2T physics. The $d+2$ dimensional spaces given by (3.24) and (3.26) are also possible spaces. Although these three spaces are indistinguishable at the classical level, this situation may change in quantum mechanics because the correct quantum dynamics may emerge only after these underlying metric structures in position and momentum spaces, together with the vector field of vanishing strength tensor, are explicitly taken into account in all the relevant equations.

5 Concluding Remarks

In this paper we showed that it is possible to construct, in the $d+2$ dimensional space-time of classical 2T physics, the same geometrical and topological structures that are present in the most general configuration space formulation of quantum mechanics containing gravity in d dimensions. The geometric structure is defined by a symmetric Riemannian metric tensor and the topological structure is defined by a vector field with a vanishing antisymmetric strength tensor which defines a section of a flat $U(1)$ bundle over space-time. This $d+2$ dimensional construction is possible first because of the existence of a finite local scale invariance of the 2T canonical Hamiltonian, and second because 2T physics contains at the classical level a local continuous generalization of the discrete duality symmetry between position and momentum that underlies the structure of quantum mechanics.

One of the results of this paper that requires a deeper investigation is the fact that the classical Hamiltonian 2T dynamics in the presence of the topological vector field and described by the variables X_M, P_M and $A_M(X)$ satisfying the brackets

$$\begin{aligned}
&\{P_M, P_N\} = 0,\\
&\{X_M, P_N\} = \eta_{MN},\\
&\{X_M, X_N\} = 0,\\
&\{X_M, A_N\} = 0,\\
&\{P_M, A_N\} = -\frac{\partial A_N}{\partial X^M},\\
&\{A_M, A_N\} = 0
\end{aligned}$$

is the same classical Hamiltonian dynamics described by the same variables but satisfying the brackets

$$\begin{aligned}
&\{P_M, P_N\} = L_{MN},\\
&\{X_M, P_N\} = \eta_{MN} + X_M X_N,\\
&\{X_M, X_N\} = 0,\\
&\{X_M, A_N\} = 0,\\
&\{P_M, A_N\} = -\frac{\partial A_N}{\partial X^M} + X_M A_N,\\
&\{A_M, A_N\} = 0
\end{aligned}$$

and is also the same classical Hamiltonian dynamics described by the same variables but now satisfying the brackets

$$\begin{aligned}
&\{P_M, P_N\} = 0,\\
&\{X_M, P_N\} = \eta_{MN} - P_M P_N,\\
&\{X_M, X_N\} = -L_{MN},\\
&\{X_M, A_N\} = -P_M A_N - X_M P^S \frac{\partial A_N}{\partial X^S},\\
&\{P_M, A_N\} = -\frac{\partial A_N}{\partial X^M} + P_M P^S \frac{\partial A_N}{\partial X^S},\\
&\{A_M, A_N\} = A_M P^S \frac{\partial A_N}{\partial X^S} - A_N P^S \frac{\partial A_M}{\partial X^S}.
\end{aligned}$$

We may say that, as a consequence of finite local scale invariance, three formulations of quantum dynamics in three different spaces have the same classical Hamiltonian limit described by 2T physics.

Inspired by the example of the spherical harmonic oscillator in a punctured plane described in [1], we are inclined to look at the holonomy parametrized by the 1-form $dX^M A_M$ as a higher dimensional Aharonov-Bohm flux line [26] piercing the configuration space at its origin and in whose vector potential the $d+2$ dimensional massless particle moves. This can explain the noncommutativity of the momenta that will be induced in the quantized theory if bracket (3.23a) is assumed. It is well known [27] that in a magnetic field the momenta fail to mutually commute. However, the vector field considered in this paper should not necessarily be interpreted as having an electrodynamic origin because no electric charge was assumed for the particle. Also, as became clear in the ADM construction [28] of general relativity, a neutral scalar massless relativistic particle couples only to the gravitational field. A gravitational or topological interpretation for the vector field is then also possible. An exotic but interesting possibility may be to interpret the vector field as having a gravitodynamic origin [29]. In any case, it turns out that in $d+2$ dimensions the nontrivial holonomies associated to the nontrivial representations of the Heisenberg algebra can also be regarded as being due to some specific Aharonov-Bohm flux lines passing through holes in configuration space, and which are characterized by the first homotopy group $\pi_1(M)$ of that space.

To conclude we would like to mention that the main motivation for this paper is to try to apply the ideas of 2T physics in gravitational and quantum mechanical physics. This area of theoretical physics seems to be left rather unexplored by the main researchers in 2T physics. However, it is now clear that the Standard Model of Particles and Forces in $3+1$ dimensions is only part of a "master 2T theory" [31] in $4+2$ dimensions. This "master 2T theory" is exactly the massless particle in flat $d+2$ dimensions described by action (3.5) in the case when $d=4$. The results of this paper then have the potential to bring with them entirely new ways of incorporating gravitational and topological effects into the Standard Model. As emphasized in [31], the higher space in $d+2$ dimensions is not just formalism that could be avoided. 1T physics can be used to verify and interpret the predictions of 2T physics, but it is not equipped to come up with the predictions in the first place [31], unless one stumbles into some of them occasionally, such as the $SO(d,2)$ conformal symmetry of the massless scalar relativistic particle we considered in section two. The results in this paper are then relevant because they teach us that one of the great advantages of 2T physics over 1T physics is the classical continuous local indistinguishability of position and momenta it explicitly displays. Before the advent of 2T physics this kind of indistinguishability, but in a much more restricted discontinuous global form, was long known to exist in quantum mechanics as a consequence of the wave-particle duality of matter and energy. The lessons from 2T physics so far makes it evident that the ordinary 1T physics formulation of Nature is insufficient to provide the explanation or even the existence [31] of the many unifying facts revealed through 2T physics.

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